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On the typical level crossing time and path ☆

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Abstract

Let $Y_1, Y_2, \ldots$ be a stochastic process and M a positive real number. Define the level crossing time $T_M = \inf\{n \mid Y_n > M\}$ ($T_M = +\infty$ if $Y_n \leqslant M$ for $n = 1, 2, \ldots$). We study the process with the condition that the high level M is crossed. Using the techniques of large deviations theory we describe roughly when and how the level crossing typically occurs. The main hypotheses required are stated in terms of the generating functions associated with the process (Y_n).

Keywords: Level crossing; Large deviations theory; Law of large numbers

1. Introduction

Let $Y_1, Y_2, \ldots$ be a stochastic process and M a positive real number. Define the level crossing time $T = T_M$ by $T = \inf\{n \mid Y_n > M\}$ ($T = +\infty$, if $Y_n \leqslant M$ for $n = 1, 2, \ldots$). We study the process under the condition that the crossing of the high level M occurs. We will show that, conditionally, a certain transform of T is close to a constant with a high probability when M tends to infinity. In addition, we prove that a transform of the process (Y_n) is close to a deterministic path in the sense mentioned above. These are the main results of the paper describing roughly when and how the level crossing typically occurs. They are stated in Theorems 4 and 6. Using the techniques of large deviations theory, we give sufficient conditions of a general nature for the results mentioned. The hypotheses required are stated in terms of generating functions associated with the process (Y_n).

In insurance mathematics, T can be interpreted as the time of ruin and is thus an object of interest. For processes with independent and identically distributed increments and for Markov additive processes, accurate asymptotic results are available about ruin probabilities, see e.g. Cramér (1955), Siegmund (1975), Lalley (1984) and Asmussen (1989). The goal of this paper is to prove rough limit results, which, on the other hand, are valid for more general processes. The problem of the typical time of

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ruin is explicitely studied in Martin-Löf (1986) and Embrechts et al. (1993). Theorem 4 below may be viewed as an extension of their results although the standpoint in the papers mentioned is in continuous-time processes. See also Berg (1994). The typical path to ruin explains the event in more details. Such a path is introduced by Martin-Löf (1983, 1989) in the case of random walk. For similar viewpoints in a Markovian environment, see Bucklew (1990, Ch. 4). Theorem 6 below specifies the nature of the path and generalizes the results mentioned.

In connection with ruin probabilities, it is often assumed that the process has a drift to $-\infty$. We study also processes drifting to $+\infty$. In our framework, the crossing of any positive level M is then almost sure, which simplifies examination.

Knowledge about the typical level crossing path may be useful also when problems of two levels are considered. One may then be interested in the probability that the process crosses the level M before the level $-aM$ with a positive constant a. Rough exponential estimates for these probabilities can be deduced if the typical path for the crossing of the level M does not indicate the crossing of the level $-aM$. These types of estimates may be sufficient when level crossing probabilities are estimated by simulation, see Lehtonen and Nyrhinen (1992a, b). Siegmund (1976) studied simulation problems in the case of two levels in connection with sequential tests.

In large deviations theory, the typical behavior of a subject concerned is often required as the first knowledge for large deviations limit theorems but it is also interesting as such. For our purposes, the most useful results of the theory are the general laws of large numbers presented by Ellis (1984) and improvements of Nummelin (1986, 1990).

2. Preliminaries

We present in this section some properties of convex functions and connections to large deviations theory. A function $f: \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is *convex* if the epigraph $\{(x, y) \in \mathbb{R}^2 \mid y \geqslant f(x)\}$ is a convex subset of $\mathbb{R}^2$. It is *proper convex* if, in addition, $f(t) > -\infty$ for every $t \in \mathbb{R}$ and $f(t) < \infty$ for some $t \in \mathbb{R}$. Equivalently, a function $f: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper convex, if $f(t) < \infty$ for some $t \in \mathbb{R}$ and

$$f(\alpha t + (1-\alpha)u) \leqslant \alpha f(t) + (1-\alpha) f(u)$$

for every $t, u \in \mathbb{R}$ and $\alpha \in [0, 1]$. A sufficient condition for a convex function f to be proper convex is that $f(t)$ is bounded from above for every t in a non-empty open set and finite for some t in that set. Denote

$$\mathscr{D}(f) = \{t \in \mathbb{R} \mid f(t) < \infty\}$$

and let $\dot{\mathscr{D}}(f)$ be the interior of $\mathscr{D}(f)$. For $t \in \dot{\mathscr{D}}(f)$, a proper convex function is continuous at point t and the right derivative $f'(t+)$

$$f'(t+) = \lim_{h \to 0+} h^{-1}[f(t+h) - f(t)]$$

and the left derivative $f'(t-)$ exist and are finite with $f'(t-) \leqslant f'(t+)$.

We list in Lemma 1 some more specific properties of convex functions.

Lemma 1. *Let $f_1, f_2, \ldots$ be a sequence of proper convex functions and define the function $f: \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ by*

$$f(t) = \limsup_{n\to\infty} f_n(t).$$

Assume that for a fixed $t_0 \in \mathbb{R}$ and for every $n \in \mathbb{N}$, $f_n(t_0)$ equals a constant $a \in \mathbb{R}$. Suppose that there exists a function $F: \mathbb{R} \to \mathbb{R}$ such that $F(t_0) = a$ and $f(t) \leqslant F(t)$ in a neighbourhood of t_0. Then f is a proper convex function. If the right derivative $F'(t_0+)$ exists then

$$f'(t_0+) \leqslant F'(t_0+) \tag{2.1}$$

and if the left derivative $F'(t_0-)$ exists then

$$f'(t_0-) \geqslant F'(t_0-). \tag{2.2}$$

Assume that the derivative $F'(t_0)$ exists. Then the derivative $f'(t_0)$ exists and equals $F'(t_0)$. Moreover,

$$\lim_{n\to\infty} f'_n(t_0+) = F'(t_0) \tag{2.3}$$

and

$$\lim_{n\to\infty} f'_n(t_0-) = F'(t_0). \tag{2.4}$$

Proof. The function f is convex as a limit superior of convex functions. The conditions $f(t_0) = a$ and $f(t) < \infty$ in a neighbourhood of t_0 imply proper convexity. The inequalities (2.1) and (2.2) are obvious.

Assume that the derivative $F'(t_0)$ exists. By the convexity, $f'(t_0-) \leqslant f'(t_0+)$ and thus by (2.1) and (2.2), $f'(t_0) = F'(t_0)$.

Consider (2.3) and (2.4). For sufficiently large n, the functions f_n are finite in a neighbourhood of t_0 and thus the right and left derivatives exist at point t_0. For a given $\varepsilon > 0$ choose $h > 0$ and n_0 such that

$$h^{-1}[F(t_0 + h) - F(t_0)] \leqslant F'(t_0) + \varepsilon$$

and

$$f_n(t_0 + h) \leqslant F(t_0 + h) + h\varepsilon$$

for every $n > n_0$. Then

$$\begin{aligned} f'_n(t_0+) &\leqslant h^{-1}[f_n(t_0 + h) - f_n(t_0)] \\ &\leqslant h^{-1}[F(t_0 + h) - F(t_0)] + \varepsilon \leqslant F'(t_0) + 2\varepsilon. \end{aligned}$$

Hence,

$$\limsup_{n\to\infty} f'_n(t_0+) \leqslant F'(t_0).$$

Similarly,

$$\liminf_{n\to\infty} f_n'(t_0 -) \geqslant F'(t_0).$$

By the convexity, $f_n'(t_0 -) \leqslant f_n'(t_0 +)$ implying (2.3) and (2.4). □

Consider now a sequence $Z_1, Z_2, \ldots$ of random variables defined on probability spaces $(\Omega_1, S_1, \boldsymbol{P}_1)$, $(\Omega_2, S_2, \boldsymbol{P}_2)$, ... respectively. Let (a_n) be a sequence of positive real numbers tending to infinity. For each $t \in \mathbb{R}$ define

$$\gamma_n(t) = a_n^{-1} \log \int_{\Omega_n} \mathrm{e}^{tZ_n(\omega)} \, \mathrm{d}\boldsymbol{P}_n(\omega)$$

and

$$\gamma(t) = \limsup_{n\to\infty} \gamma_n(t).$$

Clearly $\gamma_n(0) = 0$ for every $n \in \mathbb{N}$ and $\gamma(0) = 0$. We will deal merely with the cases where γ_n and γ are finite in a neighbourhood of the origin. Hölder's inequality implies that then γ_n and γ are proper convex functions.

The following theorem is a combination of the results of Ellis (1984) and Nummelin (1986, 1990).

Theorem 2. *Assume $\gamma(t)$ is finite for every t in a neighbourhood of the origin. Then, for each $\varepsilon > 0$ there exist constants $k_\varepsilon > 0$ and n_ε such that*

$$\boldsymbol{P}_n(Z_n/a_n \geqslant \gamma'(0+) + \varepsilon) \leqslant \mathrm{e}^{-k_\varepsilon a_n} \tag{2.5}$$

and

$$\boldsymbol{P}_n(Z_n/a_n \leqslant \gamma'(0-) - \varepsilon) \leqslant \mathrm{e}^{-k_\varepsilon a_n} \tag{2.6}$$

for every $n > n_\varepsilon$. Assume the variables Z_n are all defined on the same probability space $(\Omega, S, \boldsymbol{P})$. If the derivative $\gamma'(0)$ exists and if for each $\rho \in (0,1)$ the series

$$\sum_{n=1}^{\infty} \rho^{a_n}$$

is convergent, then

$$\boldsymbol{P}\left(\lim_{n\to\infty} Z_n/a_n = \gamma'(0)\right) = 1. \tag{2.7}$$

The method to prove for instance (2.5) is to use Chebycheff's inequality to obtain

$$\gamma_n(t) \geqslant t(\gamma'(0+) + \varepsilon) + a_n^{-1} \log \boldsymbol{P}_n(Z_n/a_n \geqslant \gamma'(0+) + \varepsilon) \tag{2.8}$$

for small $t > 0$. This leads to

$$\limsup_{n\to\infty} a_n^{-1} \log \boldsymbol{P}_n(Z_n/a_n \geqslant \gamma'(0+) + \varepsilon) \leqslant (\gamma(t) - t\gamma'(0+)) - \varepsilon t$$

and (2.5) follows by the definition of the right derivative.

In case the derivative $\gamma'(0)$ exists, Theorem 2 gives a law of large numbers for the sequence (Z_n). The almost sure convergence (2.7) is a consequence of (2.5), (2.6) and the Borel–Cantelli lemma. By Lemma 1, to prove these types of results, it suffices to find out an appropriate function F which majorizes γ in a neighbourhood of the origin.

Consider the case where we have for each n and for a fixed index set I, a family $\{Z_{n,\beta} \mid \beta \in I\}$ of random variables defined on the space $(\Omega_n, S_n, \boldsymbol{P}_n)$. Let

$$\gamma_I(t) = \limsup_{n\to\infty} \sup_{\beta\in I} a_n^{-1} \log \int_{\Omega_n} \mathrm{e}^{tZ_{n,\beta}(\omega)}\, \mathrm{d}\boldsymbol{P}_n(\omega).$$

Assume $\gamma_I(t)$ is finite in a neighbourhood of the origin. By taking first supremum over I in inequalities like (2.8), we obtain counterparts for (2.5) and (2.6)

$$\sup_{\beta\in I} \boldsymbol{P}_n(Z_{n,\beta}/a_n \geqslant \gamma_I'(0+) + \varepsilon) \leqslant \mathrm{e}^{-k_\varepsilon(I)a_n} \tag{2.9}$$

and

$$\sup_{\beta\in I} \boldsymbol{P}_n(Z_{n,\beta}/a_n \leqslant \gamma_I'(0-) - \varepsilon) \leqslant \mathrm{e}^{-k_\varepsilon(I)a_n} \tag{2.10}$$

for some $k_\varepsilon(I) > 0$ and every $n > n_\varepsilon(I)$. See also de Acosta (1985).

The results of Section 2, except (2.7), hold also for sequences of functions and random variables indexed by a real-valued parameter instead of n with obvious changes in assumptions.

For more information about convex functions and large deviations theory, we refer to Rockafellar (1970) and Ellis (1985).

3. Main results

We state in this section the limit theorems concerning level crossings. Proofs are deferred to the end of section.

Let $(\Omega, S, \boldsymbol{P})$ be a probability space and $Y_1, Y_2, \ldots$ a sequence of random variables on the measurable space (Ω, S). Let (a_n) be a sequence of positive real numbers tending to infinity.

Define the functions c_n and $\bar{c}$ by

$$c_n(t) = a_n^{-1} \log \boldsymbol{E}\{\mathrm{e}^{tY_n}\} \tag{3.1}$$

and

$$\bar{c}(t) = \limsup_{n\to\infty} c_n(t) \tag{3.2}$$

for $t \in \mathbb{R}$. Let

$$w = \sup\{t \mid \bar{c}(t) \leqslant 0\}.$$

If (3.2) holds as a limit for some $t \in \mathbb{R}$, denote

$$c(t) = \lim_{n\to\infty} c_n(t).$$

Define the function $d: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$d(t) = \sup_{n\in\mathbb{N}} c_n(t).$$

It is a proper convex function with $d(0) = 0$.

Consider now level crossing probabilities. For $M > 0$ define $T = T_M$ by

$$T = \begin{cases} \inf\{n \mid Y_n > M\}, \\ +\infty, \quad \text{if } Y_n \leqslant M \text{ for } n = 1, 2, \dots . \end{cases}$$

We state in Theorem 3 a result concerning the asymptotics of the probabilities $\boldsymbol{P}(T < \infty)$. The proof can be found in Nyrhinen (1994). The following restrictions are required on the sequence (a_n):

(A1) $\quad \lim_{n\to\infty} a_n/\log n = \infty$

and

(A2) $\quad \lim_{n\to\infty} a_n/a_{n-1} = 1.$

Theorem 3. *Assume* (A1), (A2) *and that* $w < \infty$. *Suppose that* $d(t)$ *is finite for every* $t \in (-h, w + h)$ *for some* $h > 0$. *Assume that the limit* $c(t)$ *exists and is finite for every* t *in a neighbourhood of* w *and that the derivative* $c'(w)$ *exists, being positive. Then*

$$\lim_{M\to\infty} M^{-1} \log \boldsymbol{P}(T < \infty) = -w.$$

To obtain an idea about the magnitude of T, we will show that a_T/M tends to a constant when M tends to infinity under the condition that the level crossing occurs.

Theorem 4. *Assume the hypotheses of Theorem* 3 *and let* $\mu = 1/c'(w)$. *Then for each* $\varepsilon > 0$ *there exist constants* $k_\varepsilon > 0$ *and* M_ε *such that*

$$\boldsymbol{P}(|a_T/M - \mu| \geqslant \varepsilon \mid T < \infty) \leqslant \mathrm{e}^{-k_\varepsilon M} \tag{3.3}$$

for every $M > M_\varepsilon$ *and*

$$\lim_{M\to\infty} \boldsymbol{E}\{a_T/M \mid T < \infty\} = \mu. \tag{3.4}$$

If $w = 0$ *then*

$$\boldsymbol{P}(T < \infty) = 1 \tag{3.5}$$

for each $M > 0$ *and*

$$\boldsymbol{P}\left(\lim_{M\to\infty} a_T/M = \mu\right) = 1. \tag{3.6}$$

Consider now the behaviour of the process before and after the crossing of the level M. We study again the asymptotics when M tends to infinity. For $x \in \mathbb{R}$, denote

$$b(x) = \inf\{n \mid a_n \geqslant x\}.$$

For $M > 0$ and fixed $r > 0$, define the continuous-time processes $X_M = \{X_M(\alpha) \mid 0 < \alpha \leqslant r\}$ by

$$X_M(\alpha) = Y_{b(\alpha M)}/M, \quad 0 < \alpha \leqslant r. \tag{3.7}$$

In Theorem 6 below, we will assume that the sequence (a_n) is strictly increasing. Then, for every integer n in the interval $[1, b(rM)]$, the variable Y_n has a contribution in (3.7). By Theorem 4, for sufficiently large r, the level crossing typically occurs in this interval.

We will show that, conditioned by the occurrence of the level crossing, the trajectories of X_M are with a high probability close to a deterministic path in the sense of uniform metric. To this end, we require hypotheses about the generating functions of the pairs $(Y_{b(\alpha M)}, Y_{b(\mu M)})$.

For $t, u, \in \mathbb{R}$ and $\alpha, v \geqslant 0$ denote

$$C_M(t, u; \alpha, v) = M^{-1} \log \boldsymbol{E}\{e^{tY_{b(\alpha M)} + uY_{b(vM)}}\} \tag{3.8}$$

and

$$C(t, u; \alpha, v) = \limsup_{M\to\infty} C_M(t, u; \alpha, v).$$

Denote by $B(\alpha, \delta)$ the open ball with the centre α and the radius δ, by $\bar{B}(\alpha, \delta)$ its closure and by $B_0(\alpha, \delta)$ the set $B(\alpha, \delta) \cap (0, \infty)$.

The following lemma is required to prove Theorem 6 but it also describes roughly the typical level crossing path p concerned.

Lemma 5. *Assume the hypotheses of Theorem* 3. *For* $\alpha \geqslant 0$ *and* $\delta > 0$, *the function* $H_{\alpha,\delta}: \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$

$$H_{\alpha,\delta}(t) = \limsup_{M\to\infty} \sup_{\beta \in B_0(\alpha,\delta)} C_M(t, w; \beta, \mu)$$

is proper convex, finite in a neighbourhood of the origin and $H_{\alpha,\delta}(0) = 0$. *For every* $t \in \mathbb{R}$, the limit

$$H_\alpha(t) = \lim_{\delta \to 0+} H_{\alpha,\delta}(t)$$

exists as an extended real number and defines a proper convex function H_α. *Moreover,* $H_\alpha(t)$ *is finite in a neighbourhood of the origin and* $H_\alpha(0) = 0$.

Assume the derivative $H'_\alpha(0)$ *exists. Then the partial derivative*

$$p(\alpha) \doteq \partial/\partial t[C(t, w; \alpha, \mu)]_{|t=0} \tag{3.9}$$

exists and $p(\alpha) = H'_\alpha(0)$. *Further,* $p(\alpha) < 1$ *if* $\alpha < \mu$, $p(\alpha) = 1$ *if* $\alpha = \mu$ *and, in the case of positive* w, $p(\alpha) < 1$ *if* $\alpha > \mu$.

Assume the derivative $H'_\alpha(0)$ *exists for every* $\alpha \in [0, r]$ *for* $r > 0$. *Then the function* p *defined by* (3.9) *is continuous in* $[0, r]$.

Theorem 6. *Assume the hypotheses of Theorem* 3. *Suppose that* (a_n) *is a strictly increasing sequence and let* $r > 0$.

Assume $w = 0$. *Then for each* $\varepsilon > 0$ *there exist constants* $k_\varepsilon > 0$ *and* M_ε *such that*

$$\boldsymbol{P}\left(\sup_{0<\alpha\leqslant r} |X_M(\alpha) - \alpha c'(0)| \geqslant \varepsilon\right) \leqslant \mathrm{e}^{-k_\varepsilon M} \tag{3.10}$$

for every $M > M_\varepsilon$ *and*

$$\boldsymbol{P}\left(\lim_{M\to\infty} \sup_{0<\alpha\leqslant r} |X_M(\alpha) - \alpha c'(0)| = 0\right) = 1. \tag{3.11}$$

Assume $w > 0$. *If the derivative* $H'_\alpha(0)$ *exists for every* $\alpha \in [0, r]$ *then for each* $\varepsilon > 0$ *there exist constants* $k_\varepsilon > 0$ *and* M_ε *such that*

$$\boldsymbol{P}\left(\sup_{0<\alpha\leqslant r} |X_M(\alpha) - p(\alpha)| \geqslant \varepsilon \,|\, T < \infty\right) \leqslant \mathrm{e}^{-k_\varepsilon M} \tag{3.12}$$

for every $M > M_\varepsilon$.

Proof of Theorem 4. Consider (3.3) in the case $w > 0$. By Theorem 3, $\boldsymbol{P}(T < \infty) > 0$ for every $M > 0$ and thus the conditional probabilities in (3.3) are defined. For t in a neighbourhood of w, the limit $c(t)$ exists and defines a strictly increasing function. Restricted to such a neighbourhood, c has the inverse map c^{-1} defined in a neighbourhood of the origin. Clearly, c^{-1} is positive, $c^{-1}(0) = w$ and $(c^{-1})'(0) = \mu$.

In order to apply Theorem 2, define the conditional probability spaces $(\Omega, S, \boldsymbol{P}_M)$ by

$$\boldsymbol{P}_M(B) = \boldsymbol{P}(B \,|\, T_M < \infty)$$

for $B \in S$. Let

$$Z_M = a_T 1(T < \infty),$$

where $1(A)$ denotes the indicator function of the set A: $1(A)(x) = 1$, when $x \in A$, and otherwise $1(A)(x) = 0$. We will show that

$$\limsup_{M \to \infty} M^{-1} \log \boldsymbol{E}\{e^{tZ_M} \mid T < \infty\} \leqslant w - c^{-1}(-t) \tag{3.13}$$

for every t in a neighbourhood of the origin. By Lemma 1 and Theorem 2, this suffices for (3.3).

Choose $h > 0$ and $\varepsilon > 0$ sufficiently small such that $u_\varepsilon = c^{-1}(-t-\varepsilon)$ is defined and $d(u_\varepsilon)$ is finite for every $t \in (-h, h)$. Obviously,

$$\begin{aligned} \boldsymbol{E}\{e^{tZ_M} 1(T < \infty)\} &= \sum_{n=1}^{\infty} \boldsymbol{E}\{e^{ta_T} 1(T = n)\} \\ &\leqslant \sum_{n=1}^{\infty} e^{ta_n} \boldsymbol{P}(Y_n > M). \end{aligned} \tag{3.14}$$

By Chebycheff's inequality,

$$e^{a_n c_n(u_\varepsilon)} \geqslant e^{u_\varepsilon M} \boldsymbol{P}(Y_n > M)$$

and thus

$$\boldsymbol{E}\{e^{tZ_M} 1(T < \infty)\} \leqslant e^{-u_\varepsilon M} \sum_{n=1}^{\infty} e^{(t + c_n(u_\varepsilon)) a_n}. \tag{3.15}$$

Further,

$$\lim_{n \to \infty} c_n(u_\varepsilon) = -t - \varepsilon$$

and $c_n(u_\varepsilon)$ if finite for every n. By (A1), the series in (3.15) is convergent and so

$$\limsup_{M \to \infty} M^{-1} \log \boldsymbol{E}\{e^{tZ_M} 1(T < \infty)\} \leqslant -u_\varepsilon.$$

By letting ε tend to zero and applying Theorem 3 we obtain (3.13) and further (3.3) in the case $w > 0$.

Consider (3.3) in the case $w = 0$. The inverse function c^{-1} can be defined as above in a neighbourhood of the origin. For $t < 0$ we have $c^{-1}(-t) > 0$ and inequality (3.13) follows as in the case $w > 0$. For $t > 0$, we make use of the inequalities

$$\boldsymbol{P}(T = n) \leqslant \boldsymbol{P}(Y_{n-1} \leqslant M)$$

in (3.14). For sufficiently small ε we have $u_\varepsilon = c^{-1}(-t-\varepsilon) < 0$. By Chebycheff's inequality

$$e^{a_{n-1} c_{n-1}(u_\varepsilon)} \geqslant e^{u_\varepsilon M} \boldsymbol{P}(Y_{n-1} \leqslant M)$$

for $n > 1$, and like in (3.15), we obtain

$$\boldsymbol{E}\{e^{tZ_M} 1(T < \infty)\} \leqslant e^{ta_1} + e^{-u_\varepsilon M} \sum_{n=2}^{\infty} e^{ta_n + c_{n-1}(u_\varepsilon) a_{n-1}}.$$

Here u_ε is negative and inequality (3.13) follows by (A1) and (A2) for $t > 0$. This proves (3.3) in the case $w = 0$.

The limit (3.4) is a consequence of (3.13) and Lemma 1.

For (3.5) we make use of Theorem 2 to obtain

$$\begin{aligned} \boldsymbol{P}(T < \infty) &\geqslant \limsup_{n\to\infty} \boldsymbol{P}(Y_n > M) \\ &\geqslant \limsup_{n\to\infty} \boldsymbol{P}(Y_n/a_n \geqslant c'(0)/2) = 1. \end{aligned}$$

Consider the almost sure convergence (3.6). The variable M in (3.6) can be replaced by the rational-valued variable without affecting the limit. This shows that the probability in (3.6) is defined. By (A1) and Theorem 2, for almost all $\omega \in \Omega$

$$\lim_{n\to\infty} Y_n/a_n = c'(0).$$

Consider such an ω. For given $\varepsilon > 0$ and for sufficiently large M

$$M < Y_T \leqslant a_T(1/\mu + \varepsilon)$$

and

$$M \geqslant Y_{T-1} \geqslant a_{T-1}(1/\mu - \varepsilon).$$

By (A2), this implies the convergence in (3.6) and completes the proof of Theorem 4. □

In the following proofs, the notation $o(1)$ will be used to mean a sequence of real numbers which tends to zero when M tends to infinity.

Proof of Lemma 5. Let $\alpha \geqslant 0$ and $\delta > 0$. For fixed u, α and v, the functions C_M defined by (3.8) are convex in t. Hence, $H_{\alpha,\delta}$ is convex as the limit superior of the supremum of convex functions. For fixed $t \in \mathbb{R}$, $H_{\alpha,\delta}(t)$ is monotone in δ and thus the limit $H_\alpha(t)$ exists as an extended real number. Further, H_α is convex. For $q > 1$, we have by Hölder's inequality

$$C_M(t, w; \beta, \mu) \leqslant M^{-1} \log \boldsymbol{E}\{\mathrm{e}^{tqY_{b(\beta M)}}\}^{1/q} + M^{-1} \log \boldsymbol{E}\{\mathrm{e}^{wqY_{b(\mu M)}/(q-1)}\}^{(q-1)/q}. \tag{3.16}$$

Choose large q and small $h > 0$ such that $wq/(q-1) \in \mathscr{D}(d)$ and $tq \in \mathscr{D}(d)$ for every $t \in (-h, h)$. Clearly, for $x > a_1$

$$a_{b(x)-1} < x \leqslant a_{b(x)}$$

and $a_{b(x)}$ is increasing in x. By (A2)

$$\begin{aligned} \sup_{\beta \in B_0(\alpha,\delta)} M^{-1} \log \boldsymbol{E}\{\mathrm{e}^{tqY_{b(\beta M)}}\} &\leqslant a_{b((\alpha+\delta)M)} M^{-1} |\mathrm{d}(tq)| \\ &\leqslant (1 + o(1))(\alpha + \delta) |\mathrm{d}(tq)| \end{aligned}$$

when M tends to infinity. It is seen that $C_M(t, w; \beta, \mu)$ in (3.16) is bounded from above uniformly in $\beta \in B_0(\alpha, \delta)$ and $M > 1$ for every $t \in (-h, h)$. Hence, $H_{\alpha,\delta}$ and H_α are bounded above in a neighbourhood of the origin. Clearly,

$$H_{\alpha,\delta}(0) = \limsup_{M \to \infty} M^{-1} \log E\{e^{wY_{b(\mu M)}}\}$$

$$= \limsup_{M \to \infty} a_{b(\mu M)} M^{-1} c_{b(\mu M)}(w) = 0.$$

Thus, also $H_\alpha(0) = 0$. Consequently, the functions $H_{\alpha,\delta}$ and H_α are proper convex and finite in a neighbourhood of the origin.

Assume the derivative $H'_\alpha(0)$ exists for some $\alpha \geqslant 0$. Obviously,

$$C(t, w; \alpha, \mu) \leqslant H_\alpha(t)$$

and by Lemma 1, the partial derivative $p(\alpha)$ exists and equals $H'_\alpha(0)$.

For $\alpha = \mu$ we have

$$C(t, w; \alpha, \mu) = \mu c(t + w)$$

and thus $p(\mu) = 1$.

Consider the upper bounds for $p(\alpha)$. For $\alpha > 0$, define the functions $g : \mathbb{R}^2 \to \mathbb{R} \cup \{-\infty\}$ and $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ by

$$g(t, u) = u + tp(\alpha) - C(t, u; \alpha, \mu)$$

and

$$f(t) = tp(\alpha) - \alpha c(t).$$

Then $f(t) = g(t, 0)$ and so

$$\sup_{t, u \in \mathbb{R}} g(t, u) \geqslant \sup_{t \in \mathbb{R}} f(t). \tag{3.17}$$

The partial derivatives $g_t(0, w)$ and $g_u(0, w)$ equal zero. The function g is concave implying that the global maximum of g equals $g(0, w) = w$. Further, $f(w) = wp(\alpha)$ and $f'(w) = p(\alpha) - \alpha/\mu$. These facts together with (3.17) imply the upper bounds for $p(\alpha)$ in the case $\alpha > 0$. For $\alpha = 0$, we obtain similarly

$$w \geqslant \sup_{t \in \mathscr{D}(d)} \{tp(0)\}$$

and hence $p(0) < 1$.

Assume the derivative $H'_\alpha(0)$ exists for every $\alpha \in [0, r]$. Let $\alpha \in [0, r]$, $\varepsilon > 0$ and choose t such that $H_\alpha(t)$ is finite. Let $\delta > 0$ be such that

$$\left| H_\alpha(t) - \limsup_{M \to \infty} \sup_{\beta \in B_0(\alpha, \delta)} C_M(t, w; \beta, \mu) \right| < \varepsilon.$$

Then, for $\eta \in (0, r) \cap B(\alpha, \delta/2)$

$$H_\eta(t) \leqslant H_{\eta,\delta/2}(t) \leqslant \limsup_{M \to \infty} \sup_{\beta \in B_0(\alpha,\delta)} C_M(t, w; \beta, \mu) < H_\alpha(t) + \varepsilon.$$

Consequently, in a neighbourhood of the origin

$$\limsup_{\eta \to \alpha} H_\eta(t) \leqslant H_\alpha(t).$$

Lemma 1 applies also for the limiting procedure above and hence

$$\lim_{\eta \to \alpha} H'_\eta(0) = H'_\alpha(0)$$

showing that p is continuous in $[0, r]$. □

Proof of Theorem 6. By convention, define $a_\infty = \infty$. We prove first (3.12) including the case $w = 0$. We begin by showing that for given $\varepsilon > 0$ and fixed $\alpha \in [0, r]$ there exist constants $\delta_\alpha > 0$, $k_\alpha > 0$ and M_α such that

$$\boldsymbol{P}\left(\sup_{\beta \in B_0(\alpha,\delta_\alpha) \cap (0,r]} |X_M(\beta) - p(\beta)| \geqslant \varepsilon \,|\, T < \infty \right) \leqslant e^{-k_\alpha M} \tag{3.18}$$

for every $M > M_\alpha$.

Fix small $\varepsilon' > 0$, $\delta > \alpha\varepsilon'/(\mu - \varepsilon')$ and choose $\delta_\alpha > 0$ such that

$$\{\beta\mu/v \,|\, \beta \in B_0(\alpha, \delta_\alpha), v \in \bar{B}(\mu, \varepsilon')\} \subseteq B_0(\alpha, \delta).$$

We will prove that

$$\limsup_{M \to \infty} \sup_{\beta \in B_0(\alpha,\delta_\alpha)} M^{-1} \log \boldsymbol{E}\{e^{tY_{b(\beta M)}} 1(a_T/M \in B(\mu, \varepsilon'))\}$$
$$\leqslant -w + H_{\alpha,\delta}(t) + \varepsilon' |H_{\alpha,\delta}(t)|/\mu, \tag{3.19}$$

where $H_{\alpha,\delta}$ is as in Lemma 5.

By Chebycheff's inequality, for $\beta, v \geqslant 0$

$$e^{MC_M(t,w;\beta,v)} \geqslant e^{wM} \boldsymbol{E}\{e^{tY_{b(\beta M)}} 1(Y_{b(vM)} > M)\}.$$

Consequently,

$$\boldsymbol{E}\{e^{tY_{b(\beta M)}} 1(a_T/M \in B(\mu, \varepsilon'))\} \leqslant b((\mu + \varepsilon')M) \sup_{v \in \bar{B}(\mu,\varepsilon')} \boldsymbol{E}\{e^{tY_{b(\beta M)}} 1(Y_{b(vM)} > M)\}$$

$$\leqslant b((\mu + \varepsilon')M) e^{-wM} \sup_{v \in \bar{B}(\mu,\varepsilon')} e^{MC_M(t,w;\beta,v)}. \tag{3.20}$$

By (A1)

$$\lim_{M \to \infty} M^{-1} \log b((\mu + \varepsilon')M) = 0. \tag{3.21}$$

For $M_v = vM/\mu$ we have

$$C_M(t,w;\beta,v) = C_{M_v}(t,w;\beta\mu/v,\mu)v/\mu. \tag{3.22}$$

The combination of (3.20)–(3.22) gives by the choice of δ_α

$$\sup_{\beta\in B_0(\alpha,\delta_\alpha)} M^{-1}\log \boldsymbol{E}\{\mathrm{e}^{tY_{b(\beta M)}}1(a_T/M \in B(\mu,\varepsilon'))\}$$

$$\leqslant -w + \sup_{\beta\in B_0(\alpha,\delta_\alpha),\, v\in \bar{B}(\mu,\varepsilon')} [C_{M_v}(t,w;\beta\mu/v,\mu)v/\mu] + o(1)$$

$$\leqslant -w + \sup_{L\geqslant(\mu-\varepsilon')M/\mu}\left\{\sup_{\beta\in B_0(\alpha,\delta),\, v\in\bar{B}(\mu,\varepsilon')}[C_L(t,w;\beta,\mu)v/\mu]\right\} + o(1)$$

when M tends to infinity. This implies (3.19).

By Theorems 3 and 4

$$\limsup_{M\to\infty}\sup_{\beta\in B_0(\alpha,\delta_\alpha)} M^{-1}\log\boldsymbol{E}\{\mathrm{e}^{tY_{b(\beta M)}}\,|\,a_T/M\in B(\mu,\varepsilon')\} \leqslant H_{\alpha,\delta}(t) + \varepsilon'\,|H_{\alpha,\delta}(t)|/\mu.$$

For given $\varepsilon'' > 0$, we choose sufficiently small $\varepsilon' > 0$ and make use of (2.9), (2.10) and Lemma 1 to find constants $k_1 > 0$ and M_1 such that

$$\sup_{\beta\in B_0(\alpha,\delta_\alpha)} \boldsymbol{P}(X_M(\beta) \geqslant H'_{\alpha,\delta}(0+) + \varepsilon''/2\,|\,a_T/M\in B(\mu,\varepsilon')) \leqslant \mathrm{e}^{-k_1M} \tag{3.23}$$

and

$$\sup_{\beta\in B_0(\alpha,\delta_\alpha)} \boldsymbol{P}(X_M(\beta) \leqslant H'_{\alpha,\delta}(0-) - \varepsilon''/2\,|\,a_T/M\in B(\mu,\varepsilon')) \leqslant \mathrm{e}^{-k_1M} \tag{3.24}$$

for every $M > M_1$. By Lemmata 1 and 5

$$\lim_{\delta\to 0+} H'_{\alpha,\delta}(0+) = \lim_{\delta\to 0+} H'_{\alpha,\delta}(0-) = p(\alpha).$$

By an appropriate redefinition of ε', δ and δ_α, we may assume that

$$\sup_{\beta\in B_0(\alpha,\delta_\alpha)} \boldsymbol{P}|X_M(\beta) - p(\alpha)| \geqslant \varepsilon''|a_T/M\in B(\mu,\varepsilon')) \leqslant 2\mathrm{e}^{-k_1M} \tag{3.25}$$

for every $M > M_1$. By the continuity of p, we may also assume that

$$\sup_{\beta\in B_0(\alpha,\delta_\alpha)\cap(0,r]} |p(\beta) - p(\alpha)| \leqslant \varepsilon''.$$

Hence, for $\varepsilon'' \leqslant \varepsilon/2$

$$\boldsymbol{P}\left(\sup_{\beta\in B_0(\alpha,\delta_\alpha)\cap(0,r]} |X_M(\beta) - p(\beta)| \geqslant \varepsilon|a_T/M\in B(\mu,\varepsilon')\right)$$

$$\leqslant b((\alpha+\delta_\alpha)M) \sup_{\beta\in B_0(\alpha,\delta_\alpha)} \boldsymbol{P}(|X_M(\beta) - p(\alpha)| \geqslant \varepsilon''|a_T/M\in B(\mu,\varepsilon')).$$

By (3.25) and (A1), there exist constants $k_2 > 0$ and M_2 such that

$$\boldsymbol{P}\left(\sup_{\beta \in B_0(\alpha,\delta_\alpha)\cap(0,r]} |X_M(\beta) - p(\beta)| \geqslant \varepsilon \,|\, a_T/M \in B(\mu,\varepsilon')\right) \leqslant \mathrm{e}^{-k_2 M} \tag{3.26}$$

for every $M > M_2$. Further

$$\boldsymbol{P}\left(\sup_{\beta \in B_0(\alpha,\delta_\alpha)\cap(0,r]} |X_M(\beta) - p(\beta)| \geqslant \varepsilon \,|\, T < \infty\right)$$

$$\leqslant \boldsymbol{P}\left(\sup_{\beta \in B_0(\alpha,\delta_\alpha)\cap(0,r]} |X_M(\beta) - p(\beta)| \geqslant \varepsilon \,|\, a_T/M \in B(\mu,\varepsilon')\right)$$

$$+ \boldsymbol{P}(|a_T/M - \mu| \geqslant \varepsilon' \,|\, T < \infty).$$

By (3.26) and Theorem 4, this proves (3.18).

For every $\alpha \in [0,r]$, choose δ_α, k_α and M_α as in (3.18). By the Heine–Borel theorem, we can extract a finite cover $B(\alpha_1,\delta_{\alpha_1}), \ldots, B(\alpha_m,\delta_{\alpha_m})$ of the interval $[0,r]$. Let $k_3 = \min\{k_{\alpha_1}, \ldots, k_{\alpha_m}\}$ and $M_3 = \max\{M_{\alpha_1}, \ldots, M_{\alpha_m}\}$. Then

$$\boldsymbol{P}\left(\sup_{0<\alpha\leqslant r} |X_M(\alpha) - p(\alpha)| \geqslant \varepsilon \,|\, T < \infty\right)$$

$$\leqslant m \max_{i=1}^{m} \boldsymbol{P}\left(\sup_{\beta \in B_0(\alpha_i,\delta_{\alpha_i})\cap(0,r]} |X_M(\beta) - p(\beta)| \geqslant \varepsilon \,|\, T < \infty\right) \leqslant m\mathrm{e}^{-k_3 M}$$

for every $M > M_3$. This implies (3.12).

Consider (3.10). For $\alpha > 0$ and t in a neighbourhood of the origin we have by (A2)

$$C_M(t,w;\alpha,\mu) = M^{-1}\log \boldsymbol{E}\{\mathrm{e}^{tY_{b(\alpha M)}}\} = a_{b(\alpha M)}M^{-1}c_{b(\alpha M)}(t) = \alpha c(t) + o(1)$$

when M tends to infinity. For $\delta = \alpha/2$ and $\beta \in B(\alpha,\delta)$,

$$C_M(t,w;\beta,\mu) = (\beta/\alpha)C_L(t,w;\alpha,\mu),$$

where $L = \beta M/\alpha \geqslant M/2$. Hence,

$$C_M(t,w;\beta,\mu) = \beta c(t) + o(1)$$

uniformly for $\beta \in B(\alpha,\delta)$ when M tends to infinity. Thus

$$H_\alpha(t) = \alpha c(t). \tag{3.27}$$

In the case $\alpha = 0$ we obtain for $\delta > 0$, $\beta < \delta$ and sufficiently large M

$$C_M(t,w;\beta,\mu) = a_{b(\beta M)}M^{-1}c_{b(\beta M)}(t)$$

$$\leqslant a_{b(2\delta M)-1}M^{-1}|\mathrm{d}(t)| \leqslant 2\delta|\mathrm{d}(t)|.$$

This shows that $H_0(t)$ is identically zero in a neighbourhood of the origin and so (3.27) is true also for $\alpha = 0$. Thus, (3.12) holds with $p(\alpha) = \alpha c'(0)$. By Theorem 4, this implies (3.10).

Consider the almost sure convergence (3.11). Replace M by the rational-valued variable to see that the probability in (3.11) is defined. Let $R > r$ and denote by $[M]$ the integer part of M. By (3.10) and the Borel–Cantelli lemma, for almost all $\omega \in \Omega$

$$\lim_{M\to\infty} \sup_{0<\beta\leq R} |X_{[M]}(\beta) - \beta c'(0)| = 0. \tag{3.28}$$

For $\alpha = \beta[M]/M$

$$X_M(\alpha) - \alpha c'(0) = ([M]/M)(X_{[M]}(\beta) - \beta c'(0)).$$

Hence,

$$\sup_{0<\alpha\leq r} |X_M(\alpha) - \alpha c'(0)| \leq (1 + o(1)) \sup_{0<\beta\leq R} |X_{[M]}(\beta) - \beta c'(0)|$$

when M tends to infinity. The convergence (3.28) for ω implies the convergence in (3.11) and completes the proof of Theorem 6. □

4. Examples

We study the typical path p and limit functions H_α of Theorem 6 in two examples assuming the conditions of Theorem 3. In Example 1, we find out the path p for processes associated with moving averages, see also Gerber (1982) and Burton and Dehling (1990). Example 2 shows a class of processes for which the limit functions H_α can be simplified. We will assume that w is positive and that $a_n = n$ for $n = 1, 2, \ldots$. Write Y_n in the form $Y_n = \xi_1 + \cdots + \xi_n$.

Example 1. *Partial sums of moving averages.* Let $x_1, x_2, \ldots$ be independent and identically distributed random variables and $\rho_0, \rho_1, \ldots$ a sequence of real numbers. Let $\xi_n = \sum_{i=0}^{n-1} \rho_i x_{n-i}$ be the moving average. Denote $c_x(t) = \log E\{e^{tx_1}\}$. Assume that there exists a unique $v \in (0, \infty)$ such that $c_x(v) = 0$. Suppose that $\sum_{i=0}^{\infty} \rho_i = \rho$ with $0 < \rho < \infty$ and let

$$\underline{\rho} = \inf\left\{\sum_{i=0}^{k-1} \rho_i \,\middle|\, k \in \mathbb{N}\right\}$$

and

$$\bar{\rho} = \sup\left\{\sum_{i=0}^{k-1} \rho_i \,\middle|\, k \in \mathbb{N}\right\}.$$

We assume that $0 \in \dot{\mathscr{D}}(c_x)$, $v\underline{\rho}/\rho \in \dot{\mathscr{D}}(c_x)$ and $v\bar{\rho}/\rho \in \dot{\mathscr{D}}(c_x)$. Then the conditions of Theorem 3 are satisfied with $\bar{c}(t) = c_x(\rho t)$ and $w = v/\rho$, see Example 3 in Nyrhinen

(1994). Let $\beta \in (0, \mu]$. By writing Y_n as a sum of independent terms we obtain

$$C_M(t, w; \beta, \mu) = M^{-1} \left[\sum_{k=1}^{b(\beta M)} c_x \left(t \sum_{i=0}^{b(\beta M)-k} \rho_i + w \sum_{i=0}^{b(\mu M)-k} \rho_i \right) + \sum_{k=b(\beta M)+1}^{b(\mu M)} c_x \left(w \sum_{i=0}^{b(\mu M)-k} \rho_i \right) \right].$$

This and a similar calculation for $\beta > \mu$ shows that for t close to zero

$$H_\alpha(t) = \begin{cases} \alpha c(t+w), & \text{for } \alpha \leqslant \mu, \\ \mu c(t+w) + (\alpha - \mu) c(t), & \text{for } \alpha > \mu. \end{cases}$$

Hence, result (3.12) of Theorem 6 holds with

$$p(\alpha) = \begin{cases} \alpha/\mu, & \text{for } \alpha \leqslant \mu \\ 1 + (\alpha - \mu) c'(0), & \text{for } \alpha > \mu. \end{cases}$$

Example 2. *Uniformly bounded increments.* Assume that there exists a constant $B \in \mathbb{R}$ such that

$$\boldsymbol{P}(|\xi_n| \leqslant B) = 1, \quad n = 1, 2, \ldots .$$

Then

$$\left| \sup_{\beta \in B_0(\alpha, \delta)} C_M(t, w; \beta, \mu) - C_M(t, w; \alpha, \mu) \right| \leqslant |t| \, B(2\delta + M^{-1}).$$

Thus the function H_α is reduced to

$$H_\alpha(t) = C(t, w; \alpha, \mu).$$

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